RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. FIFTH SEMESTER EXAMINATION, MARCH 2022
THIRD YEAR [BATCH 2019-22]
Measure Theory / Probability theory and Statistics

Date : 05/03/2022 **MATHEMATICS**
Time : 11am-1pm **Paper** : MADT 2 Full Marks : 50

Answer any one of the Unit (Unit 1 or Unit 2).

Unit 1: Measure Theory

Answer as many questions you can. Maximum you can obtain is 50 marks.
All symbols bear their standard meanings unless specified otherwise.

1. (a) Let $\{A_n\}$ be a sequence of subsets of $\mathbb{R}$. If [3+3]

$$A = \{x \in \mathbb{R} | x \in A_n \text{ for infinitely many } n \in \mathbb{N}\} \quad \text{and}$$
$$B = \{x \in \mathbb{R} | x \text{ is in all but finitely many } A_n\},$$

show that

$$A = \bigcap_{n=1}^{\infty}\left[\bigcup_{k=n}^{\infty} A_k\right] \quad \text{and} \quad B = \bigcup_{n=1}^{\infty}\left[\bigcap_{k=n}^{\infty} A_k\right]$$

(b) If the above sequence $\{A_n\}$ of subsets is such that each A_n is Lebesgue measurable with $m(A_1) = 1$ and $A_n \supseteq A_{n+1}, \forall n \in \mathbb{N}$, then show that A and B are both Lebesgue measurable and $m(A) = m(B) = \lim_{n\to\infty} m(A_n)$. [6]

2. Determine whether the following statements are true or false with proper justification. No marks will be awarded without proper justification. [4 x 3]

(a) Union of finitely many non-Lebesgue measurable sets can not be Lebesgue measurable.

(b) For any proper subset A of $\mathbb{R}$, we have $L^\infty(A) \subseteq L^p(A)$ for all $p > 1$.

(c) If $|f|$ is Lebesgue integrable over $A \subseteq \mathbb{R}$, then so is f.

3. Let $\{f_n\}$ be a sequence of real valued functions defined on $\mathbb{R}$ such that $f_n \in L^1(\mathbb{R}), \forall n \in \mathbb{N}$. If $f_n \to f$ uniformly then show that for every $A \subseteq \mathbb{R}$, with $m(A) < \infty$,

$$\int_A f\, dm = \lim_{n\to\infty} \int_A f_n dm$$

Will the equality hold if $m(A) = \infty$? [6+2]

4. A continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be compactly supported if there exists a compact set $A \subseteq \mathbb{R}$ such that f vanishes outside A. For such an f show that

(a) $f \in L^p(\mathbb{R}), \forall p \geq 1$. [2]

(b) for any continuous function $g : \mathbb{R} \to \mathbb{R}$, the convolution $f \circ g$ defined as

$$f \circ g(x) = \int_{y\in\mathbb{R}} f(x)g(x-y)dm(y)$$

is a well-defined real valued continuous function. Here $dm(y)$ denotes the Lebesgue integration is done with respect to the variable y. [6]

5. Let g be a Lebesgue integrable function on $A \subseteq \mathbb{R}$ and $\{f_n\}$ be a sequence of Lebesgue measurable functions such that $|f_n(x)| \leq g(x)$ a.e. on A. Show that [10]

$$\int_A \underline{\lim}\, f_n\, dm \leq \underline{\lim} \int_A f_n\, dm \leq \overline{\lim} \int_A f_n\, dm \leq \int_A \overline{\lim}\, f_n\, dm$$

6. (a) Let $\{f_n\}$ be a sequence of Lebesgue measurable functions that converge to a real valued function f a.e.on A, where $m(A) < \infty$. Show that for any $\epsilon > 0$, there exists $B \subseteq A$ with $m(B) < \epsilon$ such that $f_n \to f$ uniformly on $A \backslash B$. [5]

   (b) Let $f_n(x) = x^n, x \in [-1, 1]$. Verify the above result for $\epsilon = 0.1$. [5]

Unit 2 : Probability theory and Statistics
Group A

Answer any 5 questions. **[5 x 6 = 30 marks]**

7. Three students have identical umbrellas which they keep in some definite place while attending classes. After classes each of them selected an umbrella at random and goes home. What is the probability that at least one umbrella goes to the original owner? [6]

8. The probability of hitting a target is 0.001 for each shot. Find the probability of hitting the target with 3 or more bullets if the number of shots is 3000. [6]

9. The joint pdf of the random variables X, Y is $f_{x,y}(x, y) = \begin{cases} c(3x + y), & 0 \leq x \leq 3, 0 \leq y \leq 2 \\ 0 \quad , & \text{elsewhere} \end{cases}$.
Find $P(X + Y < 1)$ and the marginal distribution of X. [4+2]

10. Find the Moment Generating Function of the continuous distribution with pdf $f_x(x) = \frac{1}{2}x^2 e^{-x}, 0 < x < \infty$ and hence deduce the variance. [3+3]

11. If the random variables X and Y are independent and X is normal (m_x, σ_x), Y is normal (m_y, σ_y) then find the characteristic function of the Rv $Z = aX + bY$ and the distribution of $Z(a, b$ are real constants). [4+2]

12. If X and Y are two Random Variables, then prove that (a) $E[\min\{X, Y\}] \leq \min\{E(X), E(Y)\}$, (b) $E[\max\{X, Y\}] \geq \max\{E(X), E(Y)\}$. [3+3]

13. An unbiased die is thrown 900 times. Find the minimum probability that the faces multiple of 3 occur between 270 to 330 by Tchebycheff inequality. [6]

Group B

Answer any 2 questions. **[2 x 10 = 20 marks]**

14. A sample of telephone calls received daily in a certain house of 155 days is

No. of Telephone calls	3	4	5	6	7	8	9	10	11	12	13	14	Total
Frequency	8	10	15	13	18	16	20	18	13	9	7	8	155

Compute (a) sample mean, (b) sample variance, (c) coefficient of skewness. [2+3+5]

15. (a) Compute 95% confidence interval for the mean of a normal (m, σ) population using the data: $\bar{x} = 50, \sigma = 10, n = 40$. [5]

   (b) Find the best fitting line of the form $y = ax + b$ for the bivariate sample

x	1.1	1.3	2.1	3.9	6.5	8.2	10.4
y	1.42	1.90	3.75	8.01	13.62	17.32	22.13

Also find the Correlation Coefficient $\rho(x, y)$. Remark about its goodness of fit. [3+2]

16. (a) A drag is applied to 12 patients and the increment of blood pressure were recorded to be $2, 5, -2, 3, -3, 5, 6, 0, 1, -1, 2, 3$. Is it reasonable to believe that the drag has no effect for the change of blood pressure? Test at 5% level of significance, assuming the population to be normal. [5]

(b) The following table gives the number of road accident in Delhi for 90 days:

No. of accidents	2	3	4	5	6	7	8	9	10	11	12	13	14	Total
No. of days	1	2	3	8	12	15	14	11	10	6	4	3	1	90

Find at 5% confidence level wheather the number of road accident is Poissonian when the population mean is given to be 8.65. [5]

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